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Static fundamental solutions for a bi-material full-space

B. B. Guzina[a], R. Y. S. Pak[b,*]

[a]*Department of Civil Engineering, University of Minnesota, Minneapolis MN 55455-0220, U.S.A.*
[b]*Department of Civil, Environmental and Architectural Engineering, University of Colorado, Boulder CO 80309-0428, U.S.A.*

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Abstract

In this paper, the complete static response of two joined dissimilar half-spaces due to an arbitrary interior point load is derived. By means of a method of displacement potentials and integral transforms, a dual format of the solution in the form of a Hankel integral representation and algebraic closed-form expressions is presented. As illustrations, its analytical degeneration to benchmark solutions for a homogeneous medium as well as its variation under general geometric and material conditions are shown. The importance of the dual format of the bi-material solution in connection with the method of asymptotic decomposition to the development of a rigorous treatment of its dynamic counterpart is also demonstrated. © 1998 Elsevier Science Ltd. All rights reserved.

1. Introduction

The static response of a three-dimensional solid to a point load in its interior such as those given by the solutions of Kelvin (Love, 1944) and Boussinesq (1885) is fundamental in many applications. Upon integration, such Green's functions can be used to assess deformations and the stress changes in a soil medium due to distributed loads arising from construction and foundation designs. By means of appropriate integral formulations, they are also the basis for the implementation of boundary element and integral equation methods. For rigorous solutions to elastodynamic problems, the static fundamental solutions remain equally important as a tool to deal with the singular behavior of dynamic Green's functions.

For an isotropic, bi-material full-space, Rongved (1955) first derived the static fundamental solutions by means of Papkovitch functions. His work was followed by the solution of Vijayakumar and Cormack (1987) who employed matrix representations of displacements and stresses. Pan and Chou (1979) and Konguchi et al. (1990) extended the approach of Mindlin (1936) for a homogeneous isotropic half-space to treat the case of a transversely anisotropic two-phase material.

* Author to whom correspondence should be addressed. Fax: 3034927317.

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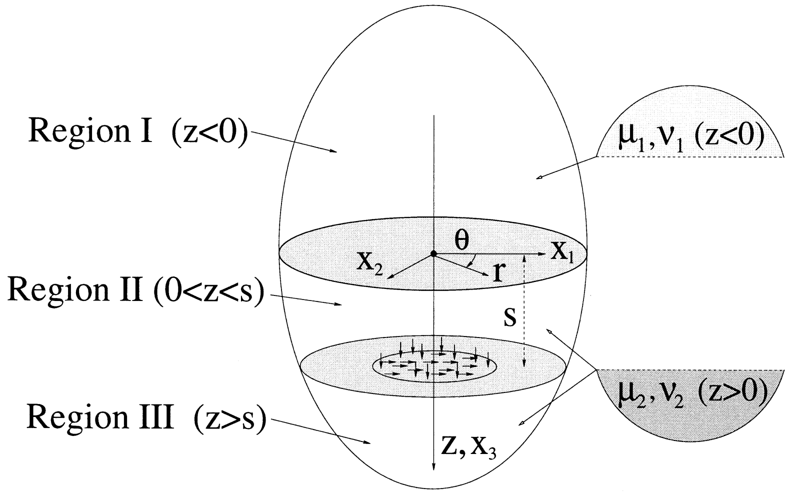

Fig. 1. Bi-material full-space.

Recently, Yu and Sanday (1991) have developed the Galerkin vectors for a number of nuclei of strain in an isotropic bi-material full-space. None of these solutions, however, is in the form of integral representations. As demonstrated in Pak (1987), Pak and Ji (1991) and Guzina and Pak (1996), the integral representations of the appropriate static fundamental solutions are necessary in the extraction of the singular behavior of various elastodynamic Green's functions derived by integral transforms. The availability of alternative representations of such singular solutions is also essential for the derivation of different forms of the boundary integral formulation (Sladek and Sladek, 1991; Tanaka et al., 1994). By means of the method of asymptotic decomposition (Pak, 1987), for instance, it was shown in Guzina (1996) that both the integral representation of static fundamental solutions for a bi-material full-space and their closed-form representation are needed for the singularity treatment of elastodynamic Green's functions for a multilayered half-space with multiple bi-material interfaces.

In what follows, the method of integral transforms and Fourier decompositions is applied in the context of displacement potentials to the point-load problem for an elastic bi-material full-space. Apart from providing new results in the form of integral representations of the displacement as well as stress Green's functions, the solution is also shown to be reducible to familiar closed-form expressions found in other treatments. The formulation and numerical implementation are illustrated by a comparison of their degenerate forms to some benchmark solutions, of which the static Green's function for a homogeneous half-space is an example. The special relevance of the dual format of the bi-material solution is demonstrated in the treatment of its dynamic counterpart for which closed-form solution is not available.

2. Formulation of the problem

In this paper, the physical domain of interest is taken to be composed of two dissimilar isotropic elastic half-spaces which are fully bonded across the plane $z = 0$ (see Fig. 1). The Lamé's constants of the upper half-space (referred to as Region I) will be denoted as λ_1 and μ_1, and the ones of the lower half-space as λ_2 and μ_2. For the derivation of the elastostatic response of a bi-material full-space to interior point loads, the displacement equilibrium equations with zero body-force field can be expressed as

$$(\lambda_b+2\mu_b)\nabla\nabla\cdot\mathbf{u}-\mu_b\nabla\times\nabla\times\mathbf{u}=\mathbf{0}, \tag{1}$$

where $\mathbf{u}$ denotes the displacement vector, and λ_b and μ_b are the piecewise constant Lamé's moduli given by

$$\lambda_b=\begin{Bmatrix}\lambda_1, & z<0\\ \lambda_2, & z>0\end{Bmatrix}, \quad \mu_b=\begin{Bmatrix}\mu_1, & z<0\\ \mu_2, & z>0\end{Bmatrix}. \tag{2}$$

Although the present formulation rules out body forces, the action of an arbitrarily distributed source across the plane $z=s$ can be represented as a set of prescribed stress discontinuities across the corresponding planar region (see Fig. 1). Without loss of generality, it is assumed that the loaded plane is located in the lower half-space, i.e. $s>0$. In the treatment of this class of problems, it is convenient to view the lower half-space as being composed of Region II $\{0<z<s\}$ and Region III $\{z>s\}$ as indicated in Fig. 1, and to represent the distributed body-force field as a general discontinuity of stresses across $z=s$ (see Pak, 1987) in cylindrical coordinates (r,θ,z), i.e.

$$\begin{aligned}\tau_{zr}(r,\theta,s^-)-\tau_{zr}(r,\theta,s^+)&=P(r,\theta),\\ \tau_{z\theta}(r,\theta,s^-)-\tau_{z\theta}(r,\theta,s^+)&=Q(r,\theta),\\ \tau_{zz}(r,\theta,s^-)-\tau_{zz}(r,\theta,s^+)&=R(r,\theta),\end{aligned} \tag{3}$$

while requiring the displacements to be continuous everywhere. As a specialization of the Somigliana–Galerkin solution to the equations of equilibrium (1), one may represent the displacement field in either material domain as

$$2\mu_b\mathbf{u}=2(1-v_b)\nabla^2\{\Phi\mathbf{e}_z\}-\nabla\nabla\cdot\{\Phi\mathbf{e}_z\}+2\nabla\times\{\Psi\mathbf{e}_z\}, \tag{4}$$

where v_b is the relevant Poisson's ratio of the medium, and Φ and Ψ are the corresponding displacement potentials which satisfy the governing equations

$$\nabla^4\Phi(r,\theta,z)=0, \quad \nabla^2\Psi(r,\theta,z)=0, \tag{5}$$

(Muki, 1960). By virtue of the completeness of the angular eigenfunctions $\{e^{im\theta}\}_{m=-\infty}^{+\infty}$, $-\pi<\theta\leqslant\pi$ with respect to the class of solutions under consideration, one may express

$$\mathbf{u}(r,\theta,z)=\sum_{m=-\infty}^{\infty}\mathbf{u}_m(r,z)\,e^{im\theta},$$

$$\Phi(r,\theta,z)=\sum_{m=-\infty}^{\infty}\Phi_m(r,z)\,e^{im\theta},$$

$$\Psi(r,\theta,z)=\sum_{m=-\infty}^{\infty}\Psi_m(r,z)\,e^{im\theta},$$

$$P(r,\theta) = \sum_{m=-\infty}^{\infty} P_m(r)\, e^{im\theta}, \quad \text{etc.} \tag{6}$$

Substitution of (6) into (5) yields the governing equations for the Fourier components of displacement potentials

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{m^2}{r^2} + \frac{\partial^2}{\partial z^2}\right)^2 \Phi_m = 0,$$

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{m^2}{r^2} + \frac{\partial^2}{\partial z^2}\right)\Psi_m = 0, \tag{7}$$

where $-\infty < m < \infty$. By means of the mth order Hankel transform with respect to the radial coordinate

$$\tilde{f}^m(\xi) = \int_0^\infty f(r) r J_m(r\xi)\,\mathrm{d}r, \tag{8}$$

whose inverse may be written as

$$f(r) = \int_0^\infty \tilde{f}^m(\xi)\xi J_m(r\xi)\,\mathrm{d}\xi, \tag{9}$$

(7) can be reduced to

$$\frac{\mathrm{d}^4}{\mathrm{d}z^4}\tilde{\Phi}^m_m - 2\xi^2\frac{\mathrm{d}^2}{\mathrm{d}z^2}\tilde{\Phi}^m_m + \xi^4\tilde{\Phi}^m_m = 0,$$

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2}\tilde{\Psi}^m_m - \xi^2\tilde{\Psi}^m_m = 0. \tag{10}$$

Consistent with the regularity conditions at infinity, the relevant solutions of (10) can be written as

$$\tilde{\Phi}^m_m(\xi,z) = (C^{\mathrm{I}}_m + D^{\mathrm{I}}_m z)\, e^{\xi z},$$

$$\tilde{\Psi}^m_m(\xi,z) = F^{\mathrm{I}}_m\, e^{\xi z}, \tag{11}$$

in Region I ($-\infty < z < 0$),

$$\tilde{\Phi}^m_m(\xi,z) = (A^{\mathrm{II}}_m + B^{\mathrm{II}}_m z)\, e^{-\xi z} + (C^{\mathrm{II}}_m + D^{\mathrm{II}}_m z)\, e^{\xi z},$$

$$\tilde{\Psi}^m_m(\xi,z) = E^{\mathrm{II}}_m\, e^{-\xi z} + F^{\mathrm{II}}_m\, e^{\xi z}, \tag{12}$$

in Region II ($0 < z < s$), and

$$\tilde{\Phi}^m_m(\xi,z) = (A^{\mathrm{III}}_m + B^{\mathrm{III}}_m z)\, e^{-\xi z},$$

$$\tilde{\Psi}^m_m(\xi,z) = E^{\mathrm{III}}_m\, e^{-\xi z}, \tag{13}$$

in Region III ($s < z < \infty$). In the above, $C_m^{\rm I}(\xi)$, $D_m^{\rm I}(\xi), \ldots, E_m^{\rm III}(\xi)$ are the integration coefficients to be determined from suitable boundary conditions. With the aid of the transformed displacement-potential relations

$$
\begin{aligned}
2\mu_b \tilde{u}_{z_m}^m &= (1-2\nu_b)\frac{\mathrm{d}^2}{\mathrm{d}z^2}\tilde{\Phi}_m^m - 2(1-\nu_b)\xi^2\tilde{\Phi}_m^m,\\
2\mu_b(\tilde{u}_{r_m}^{m+1} + i\tilde{u}_{\theta_m}^{m+1}) &= \xi\left(\frac{\mathrm{d}}{\mathrm{d}z}\tilde{\Phi}_m^m + 2i\tilde{\Psi}_m^m\right),\\
2\mu_b(\tilde{u}_{r_m}^{m-1} - i\tilde{u}_{\theta_m}^{m-1}) &= -\xi\left(\frac{\mathrm{d}}{\mathrm{d}z}\tilde{\Phi}_m^m - 2i\tilde{\Psi}_m^m\right),
\end{aligned}
\tag{14}
$$

and the stress–displacement relationship

$$
\begin{aligned}
\tilde{\tau}_{zz_m}^m &= \frac{\lambda_b\xi}{2}\{(\tilde{u}_{r_m}^{m+1} + i\tilde{u}_{\theta_m}^{m+1}) - (\tilde{u}_{r_m}^{m-1} - i\tilde{u}_{\theta_m}^{m-1})\} + (\lambda_b + 2\mu_b)\frac{\mathrm{d}}{\mathrm{d}z}(\tilde{u}_{z_m}^m),\\
\tilde{\tau}_{zr_m}^{m+1} + i\tilde{\tau}_{z\theta m}^{m+1} &= \mu_b\frac{\mathrm{d}}{\mathrm{d}z}(\tilde{u}_{r_m}^{m+1} + i\tilde{u}_{\theta_m}^{m+1}) - \mu_b\xi\tilde{u}_{z_m}^m,\\
\tilde{\tau}_{zr_m}^{m-1} - i\tilde{\tau}_{z\theta_m}^{m-1} &= \mu_b\frac{\mathrm{d}}{\mathrm{d}z}(\tilde{u}_{r_m}^{m-1} - i\tilde{u}_{\theta_m}^{m-1}) + \mu_b\xi\tilde{u}_{z_m}^m,
\end{aligned}
\tag{15}
$$

$$
\begin{aligned}
\tilde{\tau}_{rr_m}^m + 2\mu_b\widetilde{\left(\frac{u_{r_m}}{r} + im\frac{\mu_{\theta_m}}{r}\right)}^m &= \frac{(\lambda_b + 2\mu_b)\xi}{2}\{(\tilde{u}_{r_m}^{m+1} + i\tilde{u}_{\theta_m}^{m+1}) - (\tilde{u}_{r_m}^{m-1} - i\tilde{u}_{\theta_m}^{m-1})\} + \lambda_b\frac{\mathrm{d}}{\mathrm{d}z}(\tilde{u}_{z_m}^m),\\
\tilde{\tau}_{\theta\theta_m}^m - 2\mu_b\widetilde{\left(\frac{u_{r_m}}{r} + im\frac{u_{\theta_m}}{r}\right)}^m &= \frac{\lambda_b\xi}{2}\{(\tilde{u}_{r_m}^{m+1} + i\tilde{u}_{\theta_m}^{m+1}) - (\tilde{u}_{r_m}^{m-1} - i\tilde{u}_{\theta_m}^{m-1})\} + \lambda_b\frac{\mathrm{d}}{\mathrm{d}z}(\tilde{u}_{z_m}^m),\\
\tilde{\tau}_{r\theta_m}^m + 2\mu_b\widetilde{\left(\frac{u_{\theta_m}}{r} - im\frac{u_{rm}}{r}\right)}^m &= \frac{-i\mu_b\xi}{2}\{(\tilde{u}_{r_m}^{m+1} + i\tilde{u}_{\theta_m}^{m+1}) + (\tilde{u}_{r_m}^{m-1} - i\tilde{u}_{\theta_m}^{m-1})\},
\end{aligned}
\tag{16}
$$

the transformed Fourier components of the displacement and stress fields may be expressed in terms of $C_m^{\rm I}, D_m^{\rm I}, \ldots, E_m^{\rm III}$ appropriate to the boundary value problem of interest.

3. Solution for a distributed buried source

By means of (11)–(16), the interfacial conditions (3) together with the continuity of displacements across the plane $z = s$ and the continuity of displacements and tractions between the two half-spaces across the plane $z = 0$ provide twelve equations required for the solution of the twelve unknown coefficients $C_m^{\rm I}, D_m^{\rm I}, \ldots, E_m^{\rm III}$. Substitution of the result into eqn (14) yields the transformed Fourier components of the displacement field in the form of

$$
\begin{aligned}
\tilde{u}_{z_m}^m &= \Omega_1(\xi, z; s)\frac{X_m - Y_m}{2\mu_2} + \Omega_2(\xi, z; s)\frac{Z_m}{\mu_2},\\
\tilde{u}_{r_m}^{m+1} + i\tilde{u}_{\theta_m}^{m+1} &= -\gamma_1(\xi, z; s)\frac{X_m - Y_m}{2\mu_2} + \gamma_2(\xi, z; s)\frac{X_m + Y_m}{2\mu_2} - \gamma_3(\xi, z; s)\frac{Z_m}{\mu_2},\\
\tilde{u}_{r_m}^{m-1} - i\tilde{u}_{\theta_m}^{m-1} &= \gamma_1(\xi, z; s)\frac{X_m - Y_m}{2\mu_2} + \gamma_2(\xi, z; s)\frac{X_m + Y_m}{2\mu_2} + \gamma_3(\xi, z; s)\frac{Z_m}{\mu_2},
\end{aligned} \tag{17}
$$

where

$$
\begin{aligned}
Z_m(\xi) &= \tilde{R}_m^m(\xi),\\
X_m(\xi) &= \tilde{P}_m^{m-1}(\xi) - i\tilde{Q}_m^{m-1}(\xi),\\
Y_m(\xi) &= \tilde{P}_m^{m+1}(\xi) + i\tilde{Q}_m^{m+1}(\xi),
\end{aligned} \tag{18}
$$

while the auxiliary functions $\Omega_1, \ldots, \gamma_3$ are defined as follows:

For Region I,

$$
\begin{aligned}
\Omega_1 &= \frac{\mu_2\, e^{-\xi d_1}}{2\xi M_1 M_2}\{\xi(zM_2 - sM_1) - (\mu_1(1-2v_1)(3-4v_2) - \mu_2(1-2v_2)(3-4v_1))\},\\
\Omega_2 &= \frac{-\mu_2\, e^{-\xi d_1}}{2\xi M_1 M_2}\{\xi(zM_2 - sM_1) - (\mu_1(2-2v_1)(3-4v_2) + \mu_2(2-2v_2)(3-4v_1))\},\\
\gamma_1 &= \frac{\mu_2\, e^{-\xi d_1}}{2\xi M_1 M_2}\{\xi(zM_2 - sM_1) + (\mu_1(2-2v_1)(3-4v_2) + \mu_2(2-2v_2)(3-4v_1))\},\\
\gamma_2 &= \frac{\mu_2\, e^{-\xi d_1}}{\xi(\mu_1 + \mu_2)},\\
\gamma_3 &= \frac{-\mu_2\, e^{-\xi d_1}}{2\xi M_1 M_2}\{\xi(zM_2 - sM_1) + (\mu_1(1-2v_1)(3-4v_2) - \mu_2(1-2v_2)(3-4v_1))\},
\end{aligned} \tag{19}
$$

where

$$
\begin{aligned}
d_1 &= |z - s|,\\
d_2 &= z + s,
\end{aligned} \tag{20}
$$

and

$$
\begin{aligned}
M_1 &= \mu_1 + (3-4v_1)\mu_2,\\
M_2 &= \mu_2 + (3-4v_2)\mu_1.
\end{aligned} \tag{21}
$$

For Regions II and III,

$$
\Omega_1 = \frac{e^{-\xi d_1}}{8\xi(1-v_2)M_1 M_2}\{\xi d_3 M_1 M_2\}
$$

$$+\frac{e^{-\xi d_2}}{8\xi(1-v_2)M_1M_2}\{2\xi^2(\mu_1-\mu_2)M_1zs-\xi(\mu_1-\mu_2)(3-4v_2)M_1d_3$$
$$-4\mu_2(1-v_2)(\mu_1(1-2v_1)(3-4v_2)-\mu_2(1-2v_2)(3-4v_1))\},$$

$$\Omega_2=\frac{e^{-\xi d_1}}{8\xi(1-v_2)M_1M_2}\{(\xi d_1+3-4v_2)M_1M_2\}$$
$$+\frac{e^{-\xi d_2}}{8\xi(1-v_2)M_1M_2}\{-2\xi^2(\mu_1-\mu_2)M_1zs-\xi(\mu_1-\mu_2)(3-4v_2)M_1d_2$$
$$-(\mu_1^2(3-4v_2)^2-\mu_2^2(3-4v_1)(5-12v_2+8v_2^2)+\mu_1\mu_2(2-4v_1)(3-4v_2)(1-2v_2))\},$$

$$\gamma_1=\frac{e^{-\xi d_1}}{8\xi(1-v_2)M_1M_2}\{(-\xi d_1+3-4v_2)M_1M_2\}$$
$$+\frac{e^{-\xi d_2}}{8\xi(1-v_2)M_1M_2}\{-2\xi^2(\mu_1-\mu_2)M_1zs+\xi(\mu_1-\mu_2)(3-4v_2)M_1d_2$$
$$-(\mu_1^2(3-4v_2)^2-\mu_2^2(3-4v_1)(5-12v_2+8v_2^2)+\mu_1\mu_2(2-4v_1)(3-4v_2)(1-2v_2))\},$$

$$\gamma_2=\frac{e^{-\xi d_1}}{2\xi}-\frac{e^{-\xi d_2}}{2\xi(\mu_1+\mu_2)}\{\mu_1-\mu_2\},$$

$$\gamma_3=\frac{e^{-\xi d_1}}{8\xi(1-v_2)M_1M_2}\{-\xi d_3M_1M_2\}$$
$$+\frac{e^{-\xi d_2}}{8\xi(1-v_2)M_1M_2}\{2\xi^2(\mu_1-\mu_2)M_1zs+\xi(\mu_1-\mu_2)(3-4v_2)M_1d_3 \tag{22}$$
$$-4\mu_2(1-v_2)(\mu_1(1-2v_1)(3-4v_2)-\mu_2(1-2v_2)(3-4v_1))\},$$

where

$$d_3=z-s. \tag{23}$$

By means of the stress–displacement relationship (15)–(16) and the displacement solution (17), the transformed stresses may be expressed as

$$\tilde{\tau}^{m}_{zz_m}=\left\{(\lambda_b+2\mu_b)\frac{\mathrm{d}\Omega_1}{\mathrm{d}z}-\lambda_b\xi\gamma_1\right\}\frac{X_m-Y_m}{2\mu_2}+\left\{(\lambda_b+2\mu_b)\frac{\mathrm{d}\Omega_2}{\mathrm{d}z}-\lambda_b\xi\gamma_3\right\}\frac{Z_m}{\mu_2},$$

$$\tilde{\tau}^{m+1}_{zr_m}+i\tilde{\tau}^{m+1}_{z\theta_m}=-\mu_b\left\{\frac{\mathrm{d}\gamma_1}{\mathrm{d}z}+\xi\Omega_1\right\}\frac{X_m-Y_m}{2\mu_2}+\mu_b\left\{\frac{\mathrm{d}\gamma_2}{\mathrm{d}z}\right\}\frac{X_m+Y_m}{2\mu_2}-\mu_b\left\{\frac{\mathrm{d}\gamma_3}{\mathrm{d}z}+\xi\Omega_2\right\}\frac{Z_m}{\mu_2},$$

$$\tilde{\tau}^{m-1}_{zr_m}-i\tilde{\tau}^{m-1}_{z\theta_m}=\mu_b\left\{\frac{\mathrm{d}\gamma_1}{\mathrm{d}z}+\xi\Omega_1\right\}\frac{X_m-Y_m}{2\mu_2}+\mu_b\left\{\frac{\mathrm{d}\gamma_2}{\mathrm{d}z}\right\}\frac{X_m+Y_m}{2\mu_2}+\mu_b\left\{\frac{\mathrm{d}\gamma_3}{\mathrm{d}z}+\xi\Omega_2\right\}\frac{Z_m}{\mu_2}, \tag{24}$$

and

$$\tilde{\tau}^m_{rr_m}+2\mu_b\left(\frac{u_{r_m}}{r}+\widetilde{im}\,\frac{u_{\theta_m}}{r}\right)^m=\left\{\lambda_b\frac{\mathrm{d}\Omega_1}{\mathrm{d}z}-(\lambda_b+2\mu_b)\xi\gamma_1\right\}\frac{X_m-Y_m}{2\mu_2}$$
$$+\left\{\lambda_b\frac{\mathrm{d}\Omega_2}{\mathrm{d}z}-(\lambda_b+2\mu_b)\xi\gamma_3\right\}\frac{Z_m}{\mu_2},$$

$$\tilde{\tau}^m_{\theta\theta_m}-2\mu_b\left(\frac{u_{r_m}}{r}+\widetilde{im}\,\frac{u_{\theta_m}}{r}\right)^m=\lambda_b\left\{\frac{\mathrm{d}\Omega_1}{\mathrm{d}z}-\xi\gamma_1\right\}\frac{X_m-Y_m}{2\mu_2}+\lambda_b\left\{\frac{\mathrm{d}\Omega_2}{\mathrm{d}z}-\xi\gamma_3\right\}\frac{Z_m}{\mu_2},$$

$$\tilde{\tau}^m_{r\theta_m}+2\mu_b\left(\frac{u_{\theta_m}}{r}-\widetilde{im}\frac{u_{rm}}{r}\right)^m=-i\mu_b\{\xi\gamma_2\}\frac{X_m+Y_m}{2\mu_2}. \tag{25}$$

In the above, the derivatives of the influence functions may be expressed as

$$\frac{\mathrm{d}\Omega_1}{\mathrm{d}z}=\frac{\mu_2\,e^{-\xi d_1}}{2M_1M_2}\{\xi(zM_2-sM_1)-(\mu_1(1-2v_1)(3-4v_2)-\mu_2(1-2v_2)(3-4v_1)-M_2)\},$$
$$\frac{\mathrm{d}\Omega_2}{\mathrm{d}z}=\frac{-\mu_2\,e^{-\xi d_1}}{2M_1M_2}\{\xi(zM_2-sM_1)-(\mu_1(2-2v_1)(3-4v_2)+\mu_2(2-2v_2)(3-4v_1)-M_2)\},$$
$$\frac{\mathrm{d}\gamma_1}{\mathrm{d}z}=\frac{\mu_2\,e^{-\xi d_1}}{2M_1M_2}\{\xi(zM_2-sM_1)+(\mu_1(2-2v_1)(3-4v_2)+\mu_2(2-2v_2)(3-4v_1)+M_2)\},$$
$$\frac{\mathrm{d}\gamma_2}{\mathrm{d}z}=\frac{\mu_2\,e^{-\xi d_1}}{\mu_1+\mu_2},$$
$$\frac{\mathrm{d}\gamma_3}{\mathrm{d}z}=\frac{-\mu_2\,e^{-\xi d_1}}{2M_1M_2}\{\xi(zM_2-sM_1)-(\mu_1(1-2v_1)(3-4v_2)-\mu_2(1-2v_2)(3-4v_1)+M_2)\}, \tag{26}$$

in Region I, and

$$\frac{\mathrm{d}\Omega_1}{\mathrm{d}z}=\frac{e^{-\xi d_1}}{8(1-v_2)M_1M_2}\{-(\xi d_1-1)M_1M_2\}$$
$$-\frac{e^{-\xi d_2}}{8(1-v_2)M_1M_2}\{(\mu_1-\mu_2)M_1(2\xi^2zs-\xi\{z(3-4v_2)-s(1-4v_2)\})$$
$$+(\mu_1^2(3-4v_2)+\mu_2^2(3-4v_1)(1-8v_2+8v_2^2)-2\mu_1\mu_2(1-2v_1)(3-4v_2)(1-2v_2))\},$$
$$\frac{\mathrm{d}\Omega_2}{\mathrm{d}z}=\frac{e^{-\xi d_1}}{8(1-v_2)M_1M_2}\{-(\xi d_1+2-4v_2)\,\mathrm{sign}(z-s)M_1M_2\}$$
$$-\frac{e^{-\xi d_2}}{8(1-v_2)M_1M_2}\{(\mu_1-\mu_2)M_1(-2\xi^2zs-\xi\{z(3-4v_2)+s(1-4v_2)\})$$

$$-(2\mu_1^2(3-4\nu_2)(1-2\nu_2)-2\mu_2^2(3-4\nu_1)(1-2\nu_2)^2-4\mu_1\mu_2\nu_2(3-4\nu_2)(1-2\nu_1))\},$$

$$\frac{\mathrm{d}\gamma_1}{\mathrm{d}z} = \frac{e^{-\xi d_1}}{8(1-\nu_2)M_1M_2}\{(\xi d_1-4+4\nu_2)\,\mathrm{sign}(z-s)M_1M_2\}$$

$$-\frac{e^{-\xi d_2}}{8(1-\nu_2)M_1M_2}\{(\mu_1-\mu_2)M_1(-2\xi^2zs+\xi\{z(3-4\nu_2)+s(5-4\nu_2)\})$$

$$-4(1-\nu_2)(\mu_1^2(3-4\nu_2)-\mu_2^2(3-4\nu_1)(2-2\nu_2)+\mu_1\mu_2(3-4\nu_2)(1-2\nu_1))\},$$

$$\frac{\mathrm{d}\gamma_2}{\mathrm{d}z} = -\frac{e^{-\xi d_1}}{2}\,\mathrm{sign}(z-s)+\frac{e^{-\xi d_2}}{2(\mu_1+\mu_2)}\{\mu_1-\mu_2\},$$

$$\frac{\mathrm{d}\gamma_3}{\mathrm{d}z} = \frac{e^{-\xi d_1}}{8(1-\nu_2)M_1M_2}\{(\xi d_1-1)M_1M_2\}$$

$$-\frac{e^{-\xi d_2}}{8(1-\nu_2)M_1M_2}\{(\mu_1-\mu_2)M_1(2\xi^2zs+\xi\{z(3-4\nu_2)-s(5-4\nu_2)\})$$

$$-(\mu_1^2(3-4\nu_2)-\mu_2^2(3-4\nu_1)(7-16\nu_2+8\nu_2^2)+2\mu_1\mu_2(1-2\nu_1)(3-4\nu_2)(3-2\nu_2))\}, \quad (27)$$

in Regions II and III.

4. Loading coefficients for concentrated point loads

For horizontal and vertical point loads, one may define the body-force fields as

$$\mathbf{f}_h(r,\theta,z) = \mathscr{F}_h\frac{\delta(r)}{2\pi r}\delta(z-s)\mathbf{e}_h,$$

$$\mathbf{f}_v(r,\theta,z) = \mathscr{F}_v\frac{\delta(r)}{2\pi r}\delta(z-s)\mathbf{e}_z. \quad (28)$$

In (28), δ is the one-dimensional Dirac delta function; $\mathbf{e}_\mathrm{h}$ is the unit horizontal vector in the $\theta = \theta_0$ direction given by

$$\mathbf{e}_h = \mathbf{e}_r\cos(\theta-\theta_0)-\mathbf{e}_\theta\sin(\theta-\theta_0), \quad (29)$$

(see also Fig. 2); $\mathbf{e}_\mathrm{r}$, $\mathbf{e}_\theta$ and $\mathbf{e}_z$ are the unit vectors in the radial, angular and vertical directions, respectively; and $\mathscr{F}_h$ and $\mathscr{F}_v$ are the force magnitudes. By virtue of the angular expansions of the stress discontinuities across the plane $z = s$ [see (3)] and the orthogonality of the angular eigenfunctions $\{e^{im\theta}\}_{m=-\infty}^{\infty}$, one finds

$$P_{\pm 1}(r) = \mathscr{F}_h e^{\mp i\theta_0}\frac{\delta(r)}{4\pi r}, \quad P_m(r) = 0, \quad m \neq \pm 1,$$

$$Q_{\pm 1}(r) = \pm i\mathscr{F}_h e^{\mp i\theta_0}\frac{\delta(r)}{4\pi r}, \quad Q_m(r) = 0, \quad m \neq \pm 1,$$

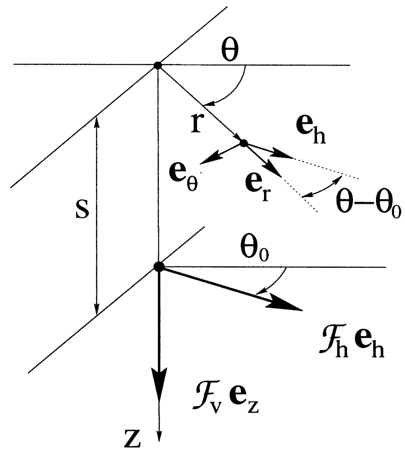

Fig. 2. Configuration of the point-loads.

$$R_0(r) = \mathscr{F}_v \frac{\delta(r)}{2\pi r}, \quad R_m(r) = 0, \quad m \neq 0, \tag{30}$$

for the point-loads in (28). They, in turn, yield the transformed loading coefficients X_m, Y_m and Z_m as

$$X_1(\xi) = \frac{\mathscr{F}_h}{2\pi} e^{-i\theta_0}, \quad X_m(\xi) = 0, \quad m \neq 1,$$

$$Y_{-1}(\xi) = \frac{\mathscr{F}_h}{2\pi} e^{i\theta_0}, \quad Y_m(\xi) = 0, \quad m \neq -1,$$

$$Z_0(r) = \frac{\mathscr{F}_v}{2\pi}, \quad Z_m(\xi) = 0, \quad m \neq 0. \tag{31}$$

5. Displacement and stress Green's functions

Upon inverting the transformed expressions (17), the displacement point-load Green's functions for the joined half-spaces may be written in cylindrical coordinates as

$$\hat{u}_r^*(r,\theta,z;s) = \frac{1}{4\pi\mu_2}\left\{-2\mathscr{F}_v \int_0^\infty (\gamma_3)\xi J_1(r\xi)\,\mathrm{d}\xi + \mathscr{F}_h \cos(\theta-\theta_0)\right.$$
$$\left.\times\left(\int_0^\infty (\gamma_2+\gamma_1)\xi J_0(r\xi)\,\mathrm{d}\xi + \int_0^\infty (\gamma_2-\gamma_1)\xi J_2(r\xi)\,\mathrm{d}\xi\right)\right\},$$

$$\hat{u}_\theta^*(r,\theta,z;s) = \frac{1}{4\pi\mu_2}\left\{-\mathscr{F}_h \sin(\theta-\theta_0)\left(\int_0^\infty (\gamma_2+\gamma_1)\xi J_0(r\xi)\,\mathrm{d}\xi - \int_0^\infty (\gamma_2-\gamma_1)\xi J_2(r\xi)\,\mathrm{d}\xi\right)\right\},$$

$$\hat{u}_z^*(r,\theta,z;s) = \frac{1}{2\pi\mu_2}\left\{\mathscr{F}_v\int_0^\infty(\Omega_2)\xi J_0(r\xi)\,\mathrm{d}\xi + \mathscr{F}_h\cos(\theta-\theta_0)\int_0^\infty(\Omega_1)\xi J_1(r\xi)\,\mathrm{d}\xi\right\}. \tag{32}$$

In the above, the symbol "$\hat{u}$" denotes the displacement Green's functions, and the superscript "*" is used to designate the direction of the point-load corresponding to the specifications in (28) and (29). Analogously, expressions (24) and (25) can be used to obtain the generalized stress Green's functions in the form of

$$\begin{aligned}
\hat{\tau}_{zz}^*(r,\theta,z;s) &= \frac{1}{2\pi\mu_2}\left\{\mathscr{F}_v\int_0^\infty\left((\lambda_b+2\mu_b)\frac{\mathrm{d}\Omega_2}{\mathrm{d}z}-\lambda_b\xi\gamma_3\right)\xi J_0(r\xi)\,\mathrm{d}\xi\right.\\
&\qquad\left.+\mathscr{F}_h\cos(\theta-\theta_0)\int_0^\infty\left((\lambda_b+2\mu_b)\frac{\mathrm{d}\Omega_1}{\mathrm{d}z}-\lambda_b\xi\gamma_1\right)\xi J_1(r\xi)\,\mathrm{d}\xi\right\},\\
\hat{\tau}_{zr}^*(r,\theta,z;s) &= \frac{\mu_b}{4\pi\mu_2}\left\{-2\mathscr{F}_v\int_0^\infty\left(\frac{\mathrm{d}\gamma_3}{\mathrm{d}z}+\xi\Omega_2\right)\xi J_1(r\xi)\,\mathrm{d}\xi+\mathscr{F}_h\cos(\theta-\theta_0)\right.\\
&\qquad\left.\times\left(\int_0^\infty\left(\frac{\mathrm{d}\gamma_2}{\mathrm{d}z}+\frac{\mathrm{d}\gamma_1}{\mathrm{d}z}+\xi\Omega_1\right)\xi J_0(r\xi)\,\mathrm{d}\xi+\int_0^\infty\left(\frac{\mathrm{d}\gamma_2}{\mathrm{d}z}-\frac{\mathrm{d}\gamma_1}{\mathrm{d}z}-\xi\Omega_1\right)\xi J_2(r\xi)\,\mathrm{d}\xi\right)\right\},\\
\hat{\tau}_{z\theta}^*(r,\theta,z;s) &= \frac{\mu_b}{4\pi\mu_2}\left\{-\mathscr{F}_h\sin(\theta-\theta_0)\right.\\
&\qquad\left.\times\left(\int_0^\infty\left(\frac{\mathrm{d}\gamma_2}{\mathrm{d}z}+\frac{\mathrm{d}\gamma_1}{\mathrm{d}z}+\xi\Omega_1\right)\xi J_0(r\xi)\,\mathrm{d}\xi-\int_0^\infty\left(\frac{\mathrm{d}\gamma_2}{\mathrm{d}z}-\frac{\mathrm{d}\gamma_1}{\mathrm{d}z}-\xi\Omega_1\right)\xi J_2(r\xi)\,\mathrm{d}\xi\right\},
\end{aligned} \tag{33}$$

$$\begin{aligned}
&\hat{\tau}_{rr}^*(r,\theta,z;s)+\frac{2\mu_b}{r}\{\hat{u}_r^*+i(\hat{u}_{\theta_1}^*e^{i\theta}-\hat{u}_{\theta_{-1}}^*e^{-i\theta})\} = \frac{1}{2\pi\mu_2}\\
&\quad\times\left\{\mathscr{F}_v\int_0^\infty\left(\lambda_b\frac{\mathrm{d}\Omega_2}{\mathrm{d}z}-(\lambda_b+2\mu_b)\xi\gamma_3\right)\xi J_0(r\xi)\,\mathrm{d}\xi\right.\\
&\quad\left.+\mathscr{F}_h\cos(\theta-\theta_0)\int_0^\infty\left(\lambda_b\frac{\mathrm{d}\Omega_1}{\mathrm{d}z}-(\lambda_b+2\mu_b)\xi\gamma_1\right)\xi J_1(r\xi)\,\mathrm{d}\xi\right\},\\
&\hat{\tau}_{\theta\theta}^*(r,\theta,z;s)-\frac{2\mu_b}{r}\{\hat{u}_r^*+i(\hat{u}_{\theta_1}^*e^{i\theta}-\hat{u}_{\theta_{-1}}^*e^{-i\theta})\} = \frac{\lambda_b}{2\pi\mu_2}\\
&\quad\times\left\{\mathscr{F}_v\int_0^\infty\left(\frac{\mathrm{d}\Omega_2}{\mathrm{d}z}-\xi\gamma_3\right)\xi J_0(r\xi)\,\mathrm{d}\xi+\mathscr{F}_h\cos(\theta-\theta_0)\int_0^\infty\left(\frac{\mathrm{d}\Omega_1}{\mathrm{d}z}-\xi\gamma_1\right)\xi J_1(r\xi)\,\mathrm{d}\xi\right\},\\
&\hat{\tau}_{r\theta}^*(r,\theta,z;s)+\frac{2\mu_b}{r}\{\hat{u}_\theta^*-i(\hat{u}_{r_1}^*e^{i\theta}-\hat{u}_{r_{-1}}^*e^{-i\theta})\} = \frac{\mu_b}{2\pi\mu_2}\left\{\mathscr{F}_h\sin(\theta-\theta_0)\int_0^\infty(\xi\gamma_2)\xi J_1(r\xi)\,\mathrm{d}\xi\right\}.
\end{aligned} \tag{34}$$

In order to evaluate the foregoing Green's functions in closed form, it is useful to denote

$$\mathcal{O}_i(n,l) = \int_0^\infty \Omega_i \xi^l J_n(r\xi)\,\mathrm{d}\xi,$$

$$\mathcal{O}_i'(n) = \int_0^\infty \frac{\mathrm{d}\Omega_i}{\mathrm{d}z} \xi J_n(r\xi)\,\mathrm{d}\xi, \quad (i=1,2)$$

$$\mathcal{G}_j(n,l) = \int_0^\infty \gamma_j \xi^l J_n(r\xi)\,\mathrm{d}\xi,$$

$$\mathcal{G}_j'(n) = \int_0^\infty \frac{\mathrm{d}\gamma_j}{\mathrm{d}z} \xi J_n(r\xi)\,\mathrm{d}\xi, \quad (j=1,2,3) \tag{35}$$

where $l = 1, 2$, $n = 0, 1, 2$ and

$$\mathcal{T}_k(n,1) \equiv \mathcal{T}(n,l,r,d_k) = \int_0^\infty e^{-\xi d_k} \xi^l J_n(r\xi)\,\mathrm{d}\xi, \quad (k=1,2) \tag{36}$$

with $l = 1, 2, 3$ and d_k given by (20). Integrals in (36) are expressible in terms of closed-form algebraic functions (Erdelyi, 1954) and may be written as

$$\mathcal{T}_k(n,l) = \left\{ \begin{array}{ll} \dfrac{(-1)^l}{r^n} \dfrac{\partial^l}{\partial d_k^l} \left\{ \dfrac{(\sqrt{d_k^2+r^2}-d_k)^n}{\sqrt{d_k^2+r^2}} \right\}, & r>0 \\ \delta_{n0}\, l!\, d_k^{-(l+1)}, & r=0 \end{array} \right\}, \tag{37}$$

where δ_{n0} is the Kronecker delta and $n > -(l+1)$. By virtue of the definitions in (19) and (26), the improper integrals in (35) may be expressed in terms of $\mathcal{T}_k(n,l)$ in Region I ($z < 0$) as

$$\mathcal{O}_1(n,l) = \frac{\mu_2}{2M_1M_2} \{ (zM_2 - sM_1)\mathcal{T}_1(n,l)$$
$$- (\mu_1(1-2v_1)(3-4v_2) - \mu_2(1-2v_2)(3-4v_1))\mathcal{T}_1(n,l-1) \},$$

$$\mathcal{O}_2(n,l) = \frac{-\mu_2}{2M_1M_2} \{ (zM_2 - sM_1)\mathcal{T}_1(n,l)$$
$$- (\mu_1(2-2v_1)(3-4v_2) + \mu_2(2-2v_2)(3-4v_1))\mathcal{T}_1(n,l-1) \},$$

$$\mathcal{G}_1(n,l) = \frac{\mu_2}{2M_1M_2} \{ (zM_2 - sM_1)\mathcal{T}_1(n,l)$$
$$+ (\mu_1(2-2v_1)(3-4v_2) + \mu_2(2-2v_2)(3-4v_1))\mathcal{T}_1(n,l-1) \},$$

$$\mathcal{G}_2(n,l) = \frac{\mu_2}{\mu_1+\mu_2} \{ \mathcal{T}_1(n,l-1) \},$$

$$\mathcal{G}_3(n,l) = \frac{-\mu_2}{2M_1M_2} \{ (zM_2 - sM_1)\mathcal{T}_1(n,l)$$

$$+(\mu_1(1-2\nu_1)(3-4\nu_2)-\mu_2(1-2\nu_2)(3-4\nu_1))\mathscr{T}_1(n,l-1)\}, \qquad (38)$$

and

$$\begin{aligned}
\mathscr{O}'_1(n) &= \frac{\mu_2}{2M_1M_2}\{(zM_2-sM_1)\mathscr{T}_1(n,2)\\
&\quad -(\mu_1(1-2\nu_1)(3-4\nu_2)-\mu_2(1-2\nu_2)(3-4\nu_1)-M_2)\mathscr{T}_1(n,1)\},\\
\mathscr{O}'_2(n) &= \frac{-\mu_2}{2M_1M_2}\{(zM_2-sM_1)\mathscr{T}_1(n,2)\\
&\quad -(\mu_1(2-2\nu_1)(3-4\nu_2)+\mu_2(2-2\nu_2)(3-4\nu_1)-M_2)\mathscr{T}_1(n,1)\},\\
\mathscr{G}'_1(n) &= \frac{\mu_2}{2M_1M_2}\{(zM_2-sM_1)\mathscr{T}_1(n,2)\\
&\quad +(\mu_1(2-2\nu_1)(3-4\nu_2)+\mu_2(2-2\nu_2)(3-4\nu_1)+M_2)\mathscr{T}_1(n,1)\},\\
\mathscr{G}'_2(n) &= \frac{\mu_2}{\mu_1+\mu_2}\{\mathscr{T}_1(n,1)\}\\
\mathscr{G}'_3(n) &= \frac{-\mu_2}{2M_1M_2}\{(zM_2-sM_1)\mathscr{T}_1(n,2)\\
&\quad +(\mu_1(1-2\nu_1)(3-4\nu_2)-\mu_2(1-2\nu_2)(3-4\nu_1)+M_2)\mathscr{T}_1(n,1)\}, \qquad (39)
\end{aligned}$$

where M_1, M_2 are given by (21). Likewise, the closed-form expressions for $\mathscr{O}_1, \mathscr{O}_2, \ldots, \mathscr{G}'_3$ in Regions II and III ($z > 0$) can be derived from (22) and (27) in the form of

$$\begin{aligned}
\mathscr{O}_1(n,l) &= \frac{1}{8(1-\nu_2)M_1M_2}\{M_1M_2d_3\mathscr{T}_1(n,l)\\
&\quad +2(\mu_1-\mu_2)M_1zs\mathscr{T}_2(n,l+1)-(\mu_1-\mu_2)(3-4\nu_2)M_1d_3\mathscr{T}_2(n,l)\\
&\quad -4\mu_2(1-\nu_2)(\mu_1(1-2\nu_1)(3-4\nu_2)-\mu_2(1-2\nu_2)(3-4\nu_1))\mathscr{T}_2(n,l-1)\},\\
\mathscr{O}_2(n,l) &= \frac{1}{8(1-\nu_2)M_1M_2}\{M_1M_2d_1\mathscr{T}_1(n,l)+(3-4\nu_2)M_1M_2\mathscr{T}_1(n,l-1)\\
&\quad -2(\mu_1-\mu_2)M_1zs\mathscr{T}_2(n,l+1)-(\mu_1-\mu_2)(3-4\nu_2)M_1d_2\mathscr{T}_2(n,l)\\
&\quad -(\mu_1^2(3-4\nu_2)^2-\mu_2^2(3-4\nu_1)(5-12\nu_2+8\nu_2^2)\\
&\quad +\mu_1\mu_2(2-4\nu_1)(3-4\nu_2)(1-2\nu_2))\mathscr{T}_2(n,l-1)\},\\
\mathscr{G}_1(n,l) &= \frac{1}{8(1-\nu_2)M_1M_2}\{-M_1M_2d_1\mathscr{T}_1(n,l)+(3-4\nu_2)M_1M_2\mathscr{T}_1(n,l-1)\\
&\quad -2(\mu_1-\mu_2)M_1zs\mathscr{T}_2(n,l+1)+(\mu_1-\mu_2)(3-4\nu_2)M_1d_2\mathscr{T}_2(n,l)\\
&\quad -(\mu_1^2(3-4\nu_2)^2-\mu_2^2(3-4\nu_1)(5-12\nu_2+8\nu_2^2)
\end{aligned}$$

$$+\mu_1\mu_2(2-4\nu_1)(3-4\nu_2)(1-2\nu_2))\mathcal{T}_2(n,l-1)\},$$

$$\mathcal{G}_2(n,l) = \frac{1}{2(\mu_1+\mu_2)}\{(\mu_1+\mu_2)\mathcal{T}_1(n,l-1)-(\mu_1-\mu_2)\mathcal{T}_2(n,l-1)\},$$

$$\mathcal{G}_3(n,l) = \frac{1}{8(1-\nu_2)M_1M_2}\{-M_1M_2d_3\mathcal{T}_1(n,l)$$
$$+2(\mu_1-\mu_2)M_1zs\mathcal{T}_2(n,l+1)+(\mu_1-\mu_2)(3-4\nu_2)M_1d_3\mathcal{T}_2(n,l)$$
$$-4\mu_2(1-\nu_2)(\mu_1(1-2\nu_1)(3-4\nu_2)-\mu_2(1-2\nu_2)(3-4\nu_1))\mathcal{T}_2(n,l-1)\}, \quad (40)$$

and

$$\mathcal{O}'_1(n) = \frac{1}{8(1-\nu_2)M_1M_2}\{-M_1M_2d_1\mathcal{T}_1(n,2)+M_1M_2\mathcal{T}_1(n,1)$$
$$-(\mu_1-\mu_2)M_1(2zs\mathcal{T}_2(n,3)-\{z(3-4\nu_2)-s(1-4\nu_2)\}\mathcal{T}_2(n,2))$$
$$-(\mu_1^2(3-4\nu_2)+\mu_2^2(3-4\nu_1)(1-8\nu_2+8\nu_2^2)$$
$$-2\mu_1\mu_2(1-2\nu_1)(3-4\nu_2)(1-2\nu_2))\mathcal{T}_2(n,1)\},$$

$$\mathcal{O}'_2(n) = \frac{1}{8(1-\nu_2)M_1M_2}\{-\mathrm{sign}(z-s)M_1M_2(d_1\mathcal{T}_1(n,2)+(2-4\nu_2)\mathcal{T}_1(n,1))$$
$$+(\mu_1-\mu_2)M_1(2zs\mathcal{T}_2(n,3)+\{z(3-4\nu_2)+s(1-4\nu_2)\}\mathcal{T}_2(n,2))$$
$$+(2\mu_1^2(3-4\nu_2)(1-2\nu_2)-2\mu_2^2(3-4\nu_1)(1-2\nu_2)^2$$
$$-4\mu_1\mu_2\nu_2(3-4\nu_2)(1-2\nu_1))\mathcal{T}_2(n,1)\},$$

$$\mathcal{G}'_1(n) = \frac{1}{8(1-\nu_2)M_1M_2}\{\mathrm{sign}(z-s)M_1M_2(d_1\mathcal{T}_1(n,2)-(4-4\nu_2)\mathcal{T}_1(n,1))$$
$$+(\mu_1-\mu_2)M_1(2zs\mathcal{T}_2(n,3)-\{z(3-4\nu_2)+s(5-4\nu_2)\}\mathcal{T}_2(n,2))$$
$$+4(1-\nu_2)(\mu_1^2(3-4\nu_2)-\mu_2^2(3-4\nu_1)(2-2\nu_2)$$
$$+\mu_1\mu_2(3-4\nu_2)(1-2\nu_1))\mathcal{T}_2(n,1)\},$$

$$\mathcal{G}'_2(n) = \frac{1}{2(\mu_1+\mu_2)}\{-\mathrm{sign}(z-s)(\mu_1+\mu_2)\mathcal{T}_1(n,1)+(\mu_1-\mu_2)\mathcal{T}_2(n,1)\},$$

$$\mathcal{G}'_3(n) = \frac{1}{8(1-\nu_2)M_1M_2}\{M_1M_2d_1\mathcal{T}_1(n,2)-M_1M_2\mathcal{T}_1(n,1)$$
$$-(\mu_1-\mu_2)M_1(2zs\mathcal{T}_2(n,3)+\{z(3-4\nu_2)-s(5-4\nu_2)\}\mathcal{T}_2(n,2))$$
$$+(\mu_1^2(3-4\nu_2)-\mu_2^2(3-4\nu_1)(7-16\nu_2+8\nu_2^2)$$
$$+2\mu_1\mu_2(1-2\nu_1)(3-4\nu_2)(3-2\nu_2))\mathcal{T}_2(n,1)\}, \quad (41)$$

In view of (38)–(41), the displacement and stress Green's functions for fully bonded dissimiliar half-spaces in (32)–(34) may be expressed in closed form as

$$\hat{u}_r^*(r,\theta,z\,;s) = \frac{1}{4\pi\mu_2}\{-2\mathcal{F}_v\mathcal{G}_3(1,1)+\mathcal{F}_h\cos(\theta-\theta_0)$$
$$\times(\mathcal{G}_2(0,1)+\mathcal{G}_1(0,1)+\mathcal{G}_2(2,1)-\mathcal{G}_1(2,1))\},$$
$$\hat{u}_\theta^*(r,\theta,z\,;s) = \frac{1}{4\pi\mu_2}\{-\mathcal{F}_h\sin(\theta-\theta_0)(\mathcal{G}_2(0,1)+\mathcal{G}_1(0,1)-\mathcal{G}_2(2,1)+\mathcal{G}_1(2,1))\},$$
$$\hat{u}_z^*(r,\theta,z\,;s) = \frac{1}{2\pi\mu_2}\{\mathcal{F}_v\mathcal{O}_2(0,1)+\mathcal{F}_h\cos(\theta-\theta_0)\mathcal{O}_1(1,1)\}, \tag{42}$$
$$\hat{\tau}_{zz}^*(r,\theta,z\,;s) = \frac{1}{2\pi\mu_2}\{\mathcal{F}_v((\lambda_b+2\mu_b)\mathcal{O}_2'(0)-\lambda_b\mathcal{G}_3(0,2))$$
$$+\mathcal{F}_h\cos(\theta-\theta_0)((\lambda_b+2\mu_b)\mathcal{O}_1'(1)-\lambda_b\mathcal{G}_1(1,2))\},$$
$$\hat{\tau}_{zr}^*(r,\theta,z\,;s) = \frac{\mu_b}{4\pi\mu_2}\{-2\mathcal{F}_v(\mathcal{G}_3'(1)+\mathcal{O}_2(1,2))$$
$$+\mathcal{F}_h\cos(\theta-\theta_0)(\mathcal{G}_2'(0)+\mathcal{G}_1'(0)+\mathcal{O}_1(0,2)+\mathcal{G}_2'(2)-\mathcal{G}_1'(2)-\mathcal{O}_1(2,2))\},$$
$$\hat{\tau}_{z\theta}^*(r,\theta,z\,;s) = \frac{-\mu_b}{4\pi\mu_2}\{\mathcal{F}_h\sin(\theta-\theta_0)(\mathcal{G}_2'(0)+\mathcal{G}_1'(0)+\mathcal{O}_1(0,2)-\mathcal{G}_2'(2)+\mathcal{G}_1'(2)+\mathcal{O}_1(2,2))\}, \tag{43}$$

and

$$\hat{\tau}_{rr}^*(r,\theta,z\,;s)+\frac{2\mu_b}{r}\{\hat{u}_r^*+i(\hat{u}_{\theta_1}^*e^{i\theta}-\hat{u}_{\theta_{-1}}^*e^{-i\theta})\} = \frac{1}{2\pi\mu_2}$$
$$\times\{\mathcal{F}_v(\lambda_b\mathcal{O}_2'(0)-(\lambda_b+2\mu_b)\mathcal{G}_3(0,2))+\mathcal{F}_h\cos(\theta-\theta_0)(\lambda_b\mathcal{O}_1'(1)-(\lambda_b+2\mu_b)\mathcal{G}_1(1,2))\},$$
$$\hat{\tau}_{\theta\theta}^*(r,\theta,z\,;s)-\frac{2\mu_b}{r}\{\hat{u}_r^*+i(\hat{u}_{\theta_1}^*e^{i\theta}-\hat{u}_{\theta_{-1}}^*e^{-i\theta})\} = \frac{\lambda_b}{2\pi\mu_2}$$
$$\times\{\mathcal{F}_v(\mathcal{O}_2'(0)-\mathcal{G}_3(0,2))+\mathcal{F}_h\cos(\theta-\theta_0)(\mathcal{O}_1'(1)-\mathcal{G}_1(1,2))\},$$
$$\hat{\tau}_{r\theta}^*(r,\theta,z\,;s)+\frac{2\mu_b}{r}\{\hat{u}_\theta^*-i(\hat{u}_{r_1}^*e^{i\theta}-\hat{u}_{r_{-1}}^*e^{-i\theta})\} = \frac{\mu_b}{2\pi\mu_2}\{\mathcal{F}_h\sin(\theta-\theta_0)\mathcal{G}_2(1,2)\}. \tag{44}$$

These functions have all been coded for computational purposes. The formulation may be completed by observing that

$$\hat{u}_\theta^*-i(\hat{u}_{r_1}^*e^{i\theta}-\hat{u}_{r_{-1}}^*e^{-i\theta}) = \frac{1}{2\pi\mu_2}\{\mathcal{F}_h\sin(\theta-\theta_0)(\mathcal{G}_2(2,1)-\mathcal{G}_1(2,1))\},$$
$$\hat{u}_r^*+i(\hat{u}_{\theta_1}^*e^{i\theta}-\hat{u}_{\theta_{-1}}^*e^{-i\theta}) = \frac{1}{2\pi\mu_2}\{-\mathcal{F}_v\mathcal{G}_3(1,1)+\mathcal{F}_h\cos(\theta-\theta_0)(\mathcal{G}_2(2,1)-\mathcal{G}_1(2,1))\}. \tag{45}$$

It should be noted that by means of (37), the radial coordinate r may be factored out in front of the expressions $\mathcal{G}_1(2,1)$, $\mathcal{G}_2(2,1)$ and $\mathcal{G}_3(1,1)$ in (38) and (40). As a result, the stress Green's functions (44) are well-defined even as $r \to 0$. One may also observe from eqns (37)–(44) that some of the point-load Green's functions are singular as the observation point approaches the source, i.e.

$$\hat{u}_i^*(r,\theta,z;s) = O\left(\frac{1}{\sqrt{r^2+(z-s)^2}}\right) \quad \text{as} \sqrt{r^2+(z-s)^2} \to 0,$$

$$\hat{\tau}_{ij}^*(r,\theta,z;s) = O\left(\frac{1}{r^2+(z-s)^2}\right) \quad \text{as} \sqrt{r^2+(z-s)^2} \to 0, \tag{46}$$

from some i and j.

6. Some limiting cases

As a check of the foregoing developments, it is useful to investigate the behavior of the functions $\Omega_1, \Omega_2, \ldots, \gamma_3$ for two limiting cases: (i) when the modulus of the "upper medium" ($z < 0$) is zero, and (ii) when the moduli of both media are equal. As it is apparent from the physics of the problem, such degenerate forms of the general formulation should correspond to the homogeneous half- and full-space solutions, respectively.

Upon setting $\mu_1 = 0$, $\mu_2 = \mu$ and $v_2 = v$, expressions (22) for the "lower" medium ($z > 0$) may be written as

$$\begin{aligned}
\Omega_1^{hs} &= \frac{1}{8(1-v)\xi}\{e^{-\xi d_1}\,\text{sign}(z-s)d_1\xi \\
&\quad + e^{-\xi d_2}(-2zs\xi^2 + (3-4v)\,\text{sign}(z-s)d_1\xi + (4-4v)(1-2v))\},\\
\Omega_2^{hs} &= \frac{1}{8(1-v)\xi}\{e^{-\xi d_1}(d_1\xi + 3-4v) + e^{-\xi d_2}(2zs\xi^2 + (3-4v)d_2\xi + (5-12v+8v^2))\},\\
\gamma_1^{hs} &= \frac{1}{8(1-v)\xi}\{e^{-\xi d_1}(-d_1\xi + 3-4v) + e^{-\xi d_2}(2zs\xi^2 - (3-4v)d_2\xi + (5-12v+8v^2))\},\\
\gamma_2^{hs} &= \frac{1}{2\xi}\{e^{-\xi d_1} + e^{-\xi d_2}\},\\
\gamma_3^{hs} &= \frac{1}{8(1-v)\xi}\{-e^{-\xi d_1}\,\text{sign}(z-s)d_1\xi \\
&\quad + e^{-\xi d_2}(-2zs\xi^2 - (3-4v)\,\text{sign}(z-s)d_1\xi + (4-4v)(1-2v))\},
\end{aligned} \tag{47}$$

which are identical to the solution of the homogeneous half-space problem (Saphores, 1989).

Analogously, setting $\mu_1 = \mu_2 = \mu$ and $v_1 = v_2 = v$ degenerates the influence functions (19) and (22) in both half-spaces ($-\infty < z < \infty$) to the common format

$$\Omega_1^{fs} = \frac{e^{-\xi d_1}}{8(1-v)}(\text{sign}(z-s)d_1), \quad \Omega_2^{fs} = \frac{e^{-\xi d_1}}{8(1-v)\xi}(d_1\xi + 3-4v),$$

$$\gamma_1^{fs} = \frac{-e^{-\xi d_1}}{8(1-\nu)\xi}(d_1\xi - 3 + 4\nu), \quad \gamma_2^{fs} = \frac{e^{-\xi d_1}}{2\xi}, \quad \gamma_3^{fs} = \frac{-e^{-\xi d_1}}{8(1-\nu)}(\mathrm{sign}(z-s)d_1), \tag{48}$$

which upon substitution into (32)–(34) yields the Kelvin's solution for a homogeneous full-space (Love, 1944).

7. Numerical results, applications and discussion

To illustrate the analytical results in previous sections, some typical Green's functions are presented in Figs 3–11 for four characteristic cases:

$$\begin{aligned}
&\text{Case I:} && \mu_1/\mu_2 = 0, && \nu_1 = 0.00, && \nu_2 = 0.30, && \text{(no upper half-space)},\\
&\text{Case II:} && \mu_1/\mu_2 = 1, && \nu_1 = 0.30, && \nu_2 = 0.30, && \text{(two equal half-spaces)},\\
&\text{Case III:} && \mu_1/\mu_2 = 10^7, && \nu_1 = 0.45, && \nu_2 = 0.30, && \text{(rigid upper half-space)},\\
&\text{Case IV:} && \mu_1/\mu_2 = 3, && \nu_1 = 0.30, && \nu_2 = 0.30, && \text{(stiffer upper half-space)}.
\end{aligned} \tag{49}$$

In the figures, the Cartesian components of the displacement Green's functions $\hat{U}_i^*(x_1, x_2, x_3, s)$ and the stress Green's functions $\hat{S}_{ij}^*(x_1, x_2, x_3, s)$ $(i, j = 1, 2, 3)$ are employed. They are related to the cylindrical components through

$$\begin{aligned}
\hat{U}_i^*(x_1, x_2, x_3; s) &= q_k^i \hat{u}_k^*(r, \theta, z; s)\\
\hat{S}_{ij}^*(x_1, x_2, x_3; s) &= q_k^i q_l^j \hat{\tau}_{kl}^*(r, \theta, z; s)
\end{aligned} \tag{50}$$

where k, $l = r$, θ, z and

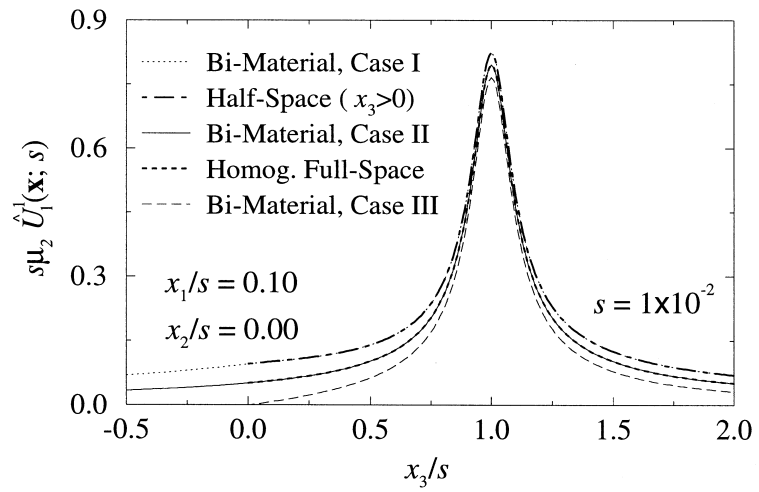

Fig. 3. Displacement Green's functions $\hat{U}_1^1(\mathbf{x}; s)$.

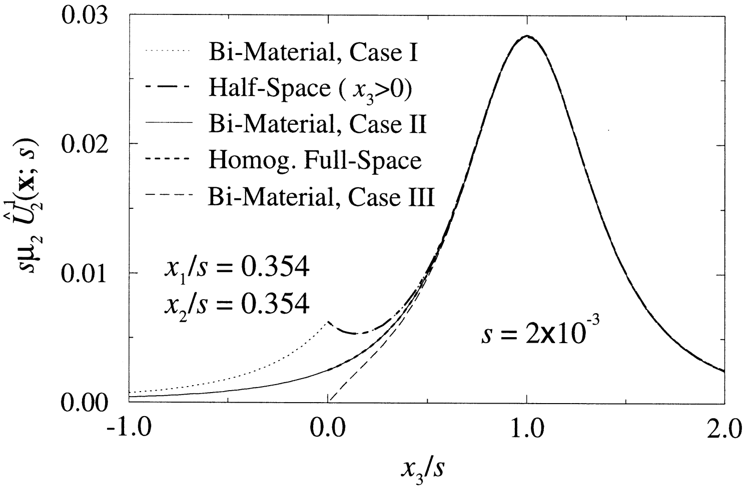

Fig. 4. Displacement Green's functions $\hat{U}_2^1(\mathbf{x}; s)$.

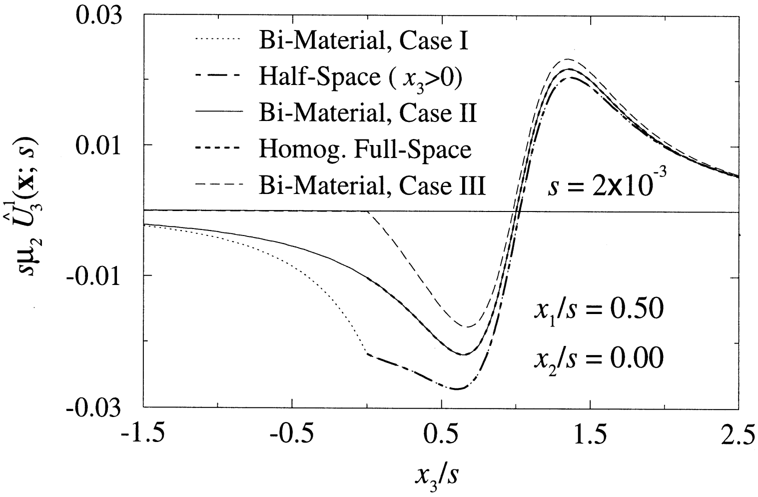

Fig. 5. Displacement Green's functions $\hat{U}_3^1(\mathbf{x}; s)$.

$$q_1^i = \frac{\partial x_i}{\partial r}, \quad q_2^i = \frac{1}{r}\frac{\partial x_i}{\partial \theta}, \quad q_3^i = \frac{\partial x_i}{\partial z}. \tag{51}$$

The first two cases in (49) are compared to the fundamental solution for a uniform half-space (Mindlin, 1936) and the Kelvin's solution for a homogeneous full-space (Love, 1944), respectively. The source point with coordinates $(0, 0, s)$ is taken to be located in the "lower" half-space, i.e. $x_3 > 0$. To emphasize the characteristic features of the bi-material formulation, the source is taken to be close to the interface $x_3 = 0$. Results are plotted along vertical lines in the z-direction.

The displacement Green's functions $\hat{U}_i^1(\mathbf{x}; s)$ $(i = 1, 2, 3)$ due to the point-load in the x_1-direction

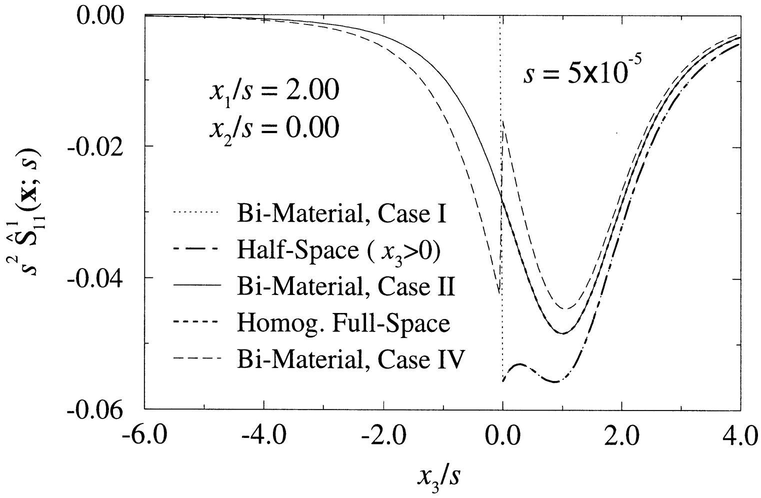

Fig. 6. Stress Green's functions $\hat{S}^1_{11}(\mathbf{x}\,;s)$.

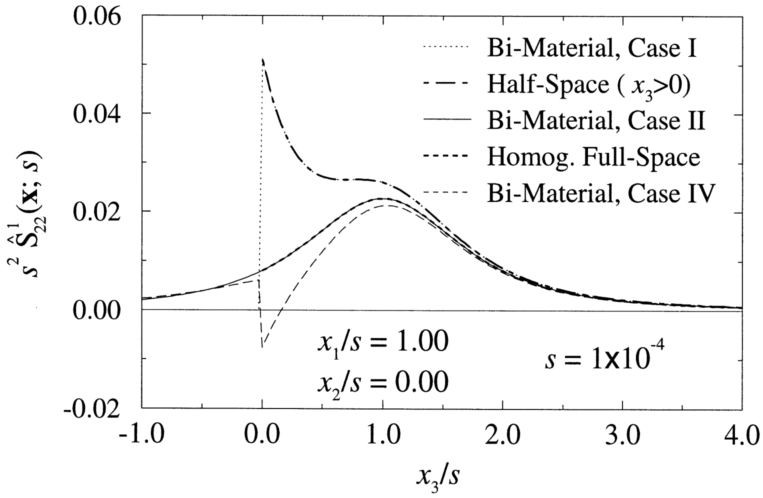

Fig. 7. Stress Green's functions $\hat{S}^1_{22}(\mathbf{x}\,;s)$.

are delineated in Figs 3–5. From the display, one may observe the full agreement of the degenerate bi-material solutions with Mindlin's and Kelvin's solution. To enhance the clarity of the plots, the benchmark solution for the full-space is presented only for $x_3 > 0$. As expected, in the case when $\mu_1/\mu_2 = 10^7$ (i.e. the "upper" medium is rigid), the displacement Green's functions vanish for $x_3 < 0$. The continuity of the displacement Green's functions across the interface $x_3 = 0$ should also be noted. While the various solutions are clearly different throughout the range in Fig. 3, the differences away from the interface are much smaller when the "secondary" effects $\hat{U}^1_2$ and $\hat{U}^1_3$ are considered in Figs 4 and 5.

The stress Green's functions $\hat{S}^1_{ij}(\mathbf{x}\,;s)$ $(i,j = 1,2,3)$ are plotted in Figs 6–11. Similar to the

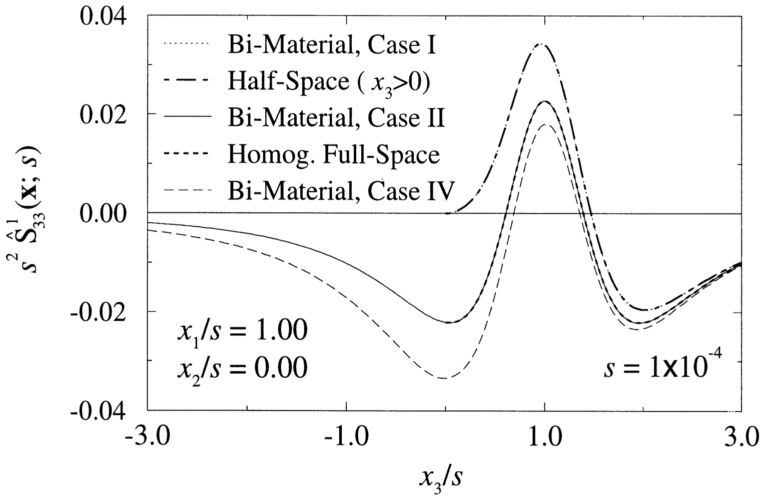

Fig. 8. Stress Green's functions $\hat{S}^1_{33}(\mathbf{x}\,;s)$.

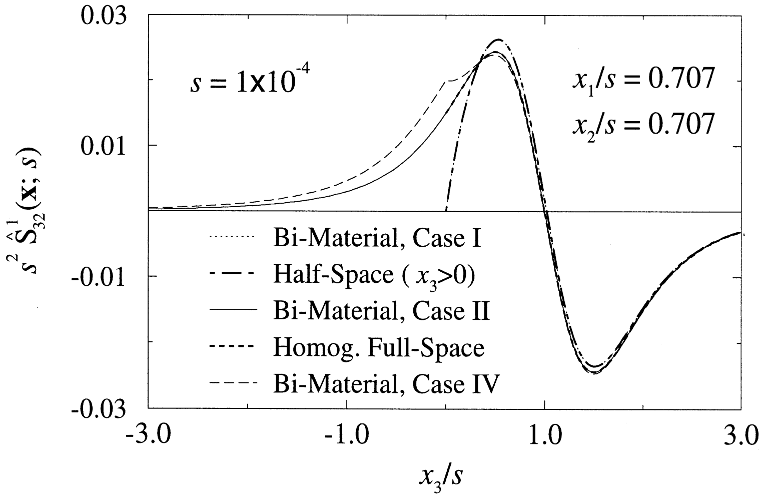

Fig. 9. Stress Green's functions $\hat{S}^1_{32}(\mathbf{x}\,;s)$.

displacement Green's functions, the stress Green's functions $\hat{S}^1_{3j}(\mathbf{x}\,;s)$ are continuous across the interface $x_3 = 0$. On the other hand, Green's functions $\hat{S}^1_{1j}$ and $\hat{S}^1_{2j}$ for $j = 1, 2$ are discontinuous at the interface whenever $\mu_1/\mu_2 \neq 1$ or $v_1/v_2 \neq 1$. Again, the Green's functions for Cases I and II are in agreement with Mindlin's and Kelvin's solution, respectively. Consistent with the symmetry of the problem, all Green's functions for the homogeneous full-space configuration (Case II) are symmetric with respect to the plane $x_3 = s$. As anticipated, effects of the neighboring medium on the stress distribution in either half-space are most pronounced close to the material interface.

Apart from its intrinsic interest as the solution of the static problem, the solution presented, because of its dual format, is also of critical importance in dealing with the more complicated

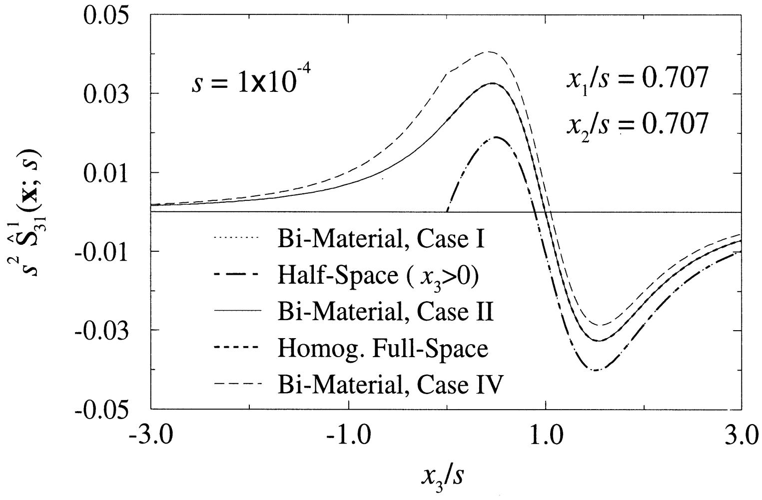

Fig. 10. Stress Green's functions $\hat{S}^1_{31}(\mathbf{x};s)$.

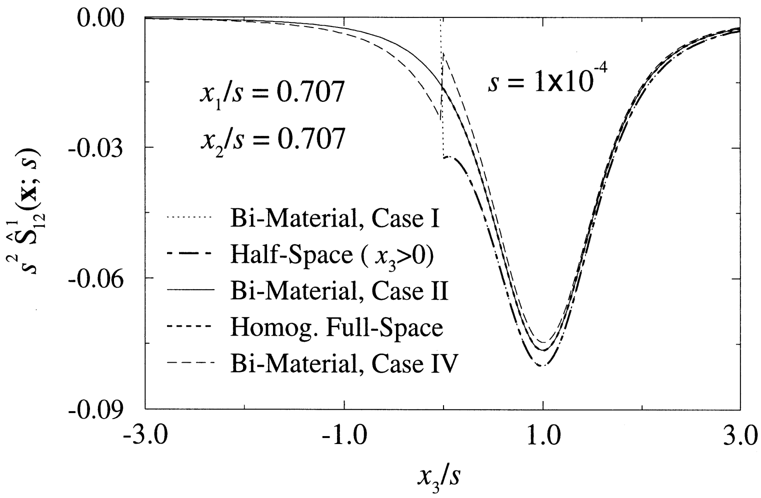

Fig. 11. Stress Green's functions $\hat{S}^1_{12}(\mathbf{x};s)$.

dynamic problem. In the context of the corresponding dynamic point-load bi-material Green's functions for instance, their analytical derivations can, to-date, only be achieved in an integral format similar to those in (32)–(34) due to the complexity of the integrands. The situation is further aggravated by the expected singular behavior of some of the Green's functions when the source approaches the receiver. To deal with such problems, it is useful to employ the method of asymptotic decomposition (Pak, 1987) wherein the leading asymptotic expansions of the featured integrals (responsible for singular behavior) are extracted and integrated analytically so that the remaining parts with strong decay can be evaluated numerically. Mathematically, one may write

$$\hat{u}_i^* = [\hat{u}_i^*]_1 + [\hat{u}_i^*]_2,$$
$$\hat{\tau}_{ij}^* = [\hat{\tau}_{ij}^*]_1 + [\hat{\tau}_{ij}^*]_2, \tag{52}$$

where the subscripts "1" and "2" denote the analytically- and numerically-evaluated parts of the fundamental solutions, respectively. For example, the time-harmonic Green's function $\hat{u}_z^*(r, \theta, z; s; \omega)$ at frequency ω for a two-phase material can be decomposed as

$$[\hat{u}_z^*(r, \theta, z; s; \omega)]_1 = \frac{1}{2\pi\mu_2}\left\{\mathscr{F}_v \int_0^\infty (\Omega_2^a)\xi J_0(r\xi)\,\mathrm{d}\xi + \mathscr{F}_h \cos(\theta-\theta_0)\int_0^\infty (\Omega_1^a)\xi J_1(r\xi)\,\mathrm{d}\xi\right\},$$

$$[\hat{u}_z^*(r, \theta, z; s; \omega)]_2 = \frac{1}{2\pi\mu_2}\left\{\mathscr{F}_v \int_0^\infty (\Omega_2^d - \Omega_2^a)\xi J_0(r\xi)\,\mathrm{d}\xi \right.$$
$$\left. + \mathscr{F}_h \cos(\theta-\theta_0)\int_0^\infty (\Omega_1^d - \Omega_1^a)\xi J_1(r\xi)\,\mathrm{d}\xi\right\} \tag{53}$$

where Ω_k^d $(k = 1, 2)$ denote the dynamic counterparts of the auxiliary functions Ω_k in (19) and (22), and

$$\Omega_1^a(\xi, z; s) = \operatorname*{Asym}_{\zeta\to\infty}\{\Omega_1^d(\xi, z; s; \omega)\},$$
$$\Omega_2^a(\xi, z; s) = \operatorname*{Asym}_{\zeta\to\infty}\{\Omega_2^d(\xi, z; s; \omega)\}, \tag{54}$$

are their leading asymptotic expansions as $\zeta \to \infty$. Here it can be shown that the asymptotic expansions $\Omega_1^a, \Omega_2^a, \ldots, \gamma_3^a$ for a two-phase material, which are independent of frequency, are identical to the static auxiliary functions Ω_1, $\Omega_2, \ldots, \gamma_3$ in (19) and (22). It was also shown in Guzina (1996) that the same asymptotic expressions are also valid in the case of dynamic Green's functions for a multi-layered half-space when the source point is near the interface of two layers with elastic properties given by $\mu_u = \mu_1$, $\nu_u = \nu_1$ for the upper stratum, and $\mu_l = \mu_2$, $\nu_l = \nu_2$ for the lower stratum.

To highlight how the static bi-material point-load solution is crucial to the treatment of the corresponding dynamic problems via the method of asymptotic decomposition, a comparison of the dynamic Green's function $\hat{U}_3^3(\mathbf{x}, s; \omega)$ for a two-phase material (Guzina, 1996) with the static solution $\hat{U}_3^3(\mathbf{x}, s)$ is shown in Fig. 12. The material properties of the bi-material full-space are given by

$$\mu_1/\mu_2 = 3, \quad \rho_1/\rho_2 = 3, \quad \nu_1 = 0.30, \quad \nu_2 = 0.20, \tag{55}$$

and the dimensionless frequency of excitation is

$$\bar{\omega} = \frac{\omega s}{\sqrt{\mu_2/\rho_2}} = 4.0. \tag{56}$$

While the infinites in Fig. 12(a) cannot be directly plotted, one can see from Fig. 12(b) that the difference between the two solutions, i.e.

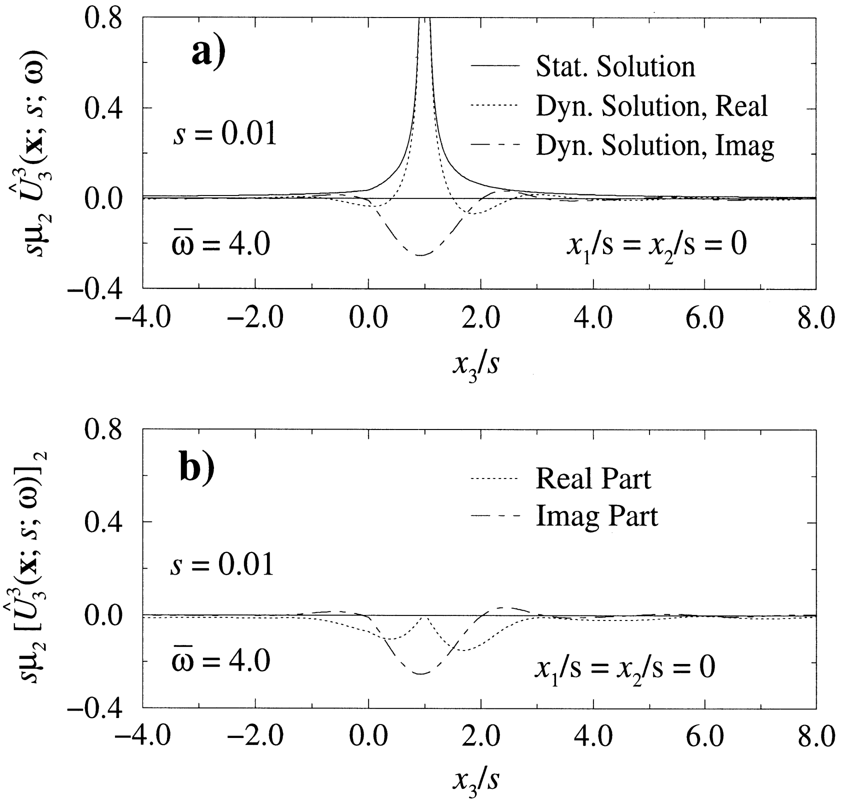

Fig. 12. Dynamic Green's function $\hat{U}_3^3(\mathbf{x}\,;s\,;\omega)$ and its regular part.

$$[\hat{U}_3^3(\mathbf{x},s\,;\omega)]_2 = \hat{U}_3^3(\mathbf{x},s\,;\omega) - [\hat{U}_3^3(\mathbf{x},s\,;\omega)]_1 \equiv \hat{U}_3^3(\mathbf{x},s\,;\omega) - \hat{U}_3^3(\mathbf{x},s), \tag{57}$$

is finite, with no singular condition anywhere.

8. Summary

In this paper, the static response of a bi-material elastic full-space due to point loads is derived by means of integral transforms and the method of displacement potentials. The solution is presented in the form of integral representations as well as in closed form, both of which are essential for the boundary element formulations and the accurate evaluation of certain elastodynamic Green's functions via the method of asymptotic decomposition. With appropriate material parameters, the bi-material solution is shown to degenerate to the solution for a uniform half-space and Kelvin's state for a homogeneous full-space. As illustrations, a set of numerical results is also presented which indicates that the effects of the neighboring medium on the deformation and the stress distribution in either half-space are most pronounced close to the material interface. As a final example, the usefulness of the dual-format static solution in the singularity extraction of the corresponding elastodynamic Green's function is demonstrated.

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